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ON THE W-GEOMETRICAL ORIGINS OF MASSLESS
 FIELD EQUATIONS AND GAUGE INVARIANCE

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ABSTRACT

We show how to obtain all covariant field equations for massless particles of arbitrary integer, or half-integer, helicity in four dimensions from the quantization of the rigid particle, whose action is given by the integrated extrinsic curvature of its worldline, *i.e.* $S = \alpha \int ds \kappa$. This geometrical particle system possesses one extra gauge invariance besides reparametrizations, and the full gauge algebra has been previously identified as classical W_3 . The key observation is that the covariantly reduced phase space of this model can be naturally identified with the spinor and twistor descriptions of the covariant phase spaces associated with massless particles of helicity $s = \alpha$. Then, standard quantization techniques require α to be quantized and show how the associated Hilbert spaces are solution spaces of the standard relativistic massless wave equations with $s = \alpha$. Therefore, providing us with a simple particle model for Weyl fermions ($\alpha = 1/2$), Maxwell fields ($\alpha = 1$), and higher spin fields. Moreover, one can go a little further and in the Maxwell case show that, after a suitable redefinition of constraints, the standard Dirac quantization procedure for first-class constraints leads to a wave-function which can be identified with the gauge potential A_μ . Gauge symmetry appears in the formalism as a consequence of the invariance under W_3 -morphisms, that is, exclusively in terms of the extrinsic geometry of paths in Minkowski space. When all gauge freedom is fixed one naturally obtains the standard Lorenz gauge condition on A_μ , and Maxwell equations in that gauge. This construction has a direct generalization to arbitrary integer values of α , and we comment on the physically interesting case of linearized Einstein gravity ($\alpha = 2$).

§1 INTRODUCTION

It was not until recently that geometrical particle models, other than the one associated with the worldline length, came to attract some attention from the physics community. And even then, they were only considered [1] as toy models for rigid strings [2], or as the simplest non-trivial examples to test the formalism of singular higher-order derivative theories [3]. Nevertheless, it was soon realized, mainly due to the pionnering work of Plyushchay, that these systems were interesting in their own right. It was shown in [4] how a noncovariant quantization of the rigid particle in Minkowski space, whose action is given by

$$S = \alpha \int ds \, \kappa, \quad (1.1)$$

where α is a dimensionless coupling constant, and the extrinsic curvature κ is given by

$$\kappa = \left| g_{\mu\nu} \frac{d^2 x^\mu}{ds^2} \frac{d^2 x^\nu}{ds^2} \right|, \quad (1.2)$$

provides us with a potential particle candidate for the description of photons and other higher order spin fields. Notwithstanding the interest of these results they fell short of proving this connection, *i.e.* it was not possible to obtain directly from this approach the associated Poincaré covariant fields theories.

After these first steps a plethora of new results have emerged in this field. Among them one should stress the ones related to Fermi-Bose transmutation in three dimensions in the presence of a Chern-Simons field. Polyakov [5] was the first to point out that the presence of a torsion term in the effective action for the Wilson loops was responsible for the appearance of Dirac fermions in an otherwise apparently bosonic theory. In particular, it was again Plyushchay [6] who realized that the Dirac equation naturally appears in a fully Lorentz covariant canonical quantization of a particle model with an extra torsion term in $2 + 1$ -dimensions (although by then there were already alternative proofs of Polyakov's results based on coherent state path integrals [7]). More recently, it was shown by the authors [8] that the extended gauge invariance present in some of these geometrical particle models could be naturally identified with the classical limit of W_n -algebras. Moreover, it was shown how the corresponding gauge transformations could be understood geometrically through the (generalized) Gauss map of their particle trajectories. Therefore providing a natural geometrical and dynamical framework for W -symmetry. In particular, for the rigid particle model (1.1) it was proven [9] that its gauge symmetry algebra could be identified with the classical limit of Zamolodchikov's W_3 -algebra. Interestingly enough the proof was based on a previously unsuspected connection with integrable systems of the KdV-type.

The purpose of this paper is to fill the gap in the results of [4] and quantize the rigid particle in a fully covariant manner. On our way we will encounter some beautiful geometric structures associated with the reduced phase space of the system under its W_3 invariance. We will show how the space of gauge invariant functions in phase space coincides with the one naturally associated with massless particles of helicity α (the coupling constant)

obtained through the coadjoint orbit method applied to the Poincaré group [10]. It is then a standard exercise in quantization to show that α is quantized and can take only integer or half integer values. Moreover, the Hilbert spaces in spinor or twistor (polarizations) coordinates are easily constructed and they are found to be, respectively, the solution spaces for the standard relativistic wave equations in spinor or twistor representation [11] with helicity α . In particular, $\alpha = 1/2, 1, 2$ correspond to the physically relevant cases of Weyl, Maxwell and linearized Einstein gravity field equations.

We will also explicitly show how to recover in the case $\alpha = 1$ the standard gauge potential (A_μ) description of Maxwell equations. This is achieved by recasting the first-class constraints of the model in spinor formalism, and quantizing them *à la* Dirac. It will then be possible to understand the standard $U(1)$ gauge symmetry of the wave function as a consequence of the W_3 gauge structure of the model, or equivalently in terms of the extrinsic geometry of paths in Minkowski space. The Lorenz gauge condition and Maxwell equations for A_μ will naturally appear from Dirac's prescription by imposing the first-class constraints as operator constraints in the wave function. This construction has a direct generalization to arbitrary integer α . We finish by commenting on its geometrical consequences in the case of linearized Einstein gravity, *i.e.*, $\alpha = 2$.

In order to be reasonably self-contained we will introduce the necessary geometrical concepts as they are needed, and will provide the reader with the minimally required knowledge about the rigid particle and its W_3 gauge invariance.

§2 THE RIGID PARTICLE

Let us briefly review some known results concerning the rigid particle model. Consider a curve γ describing the trajectory of a particle in Minkowski space

$$\begin{aligned} \gamma : [t_0, t_1] &\longrightarrow \mathbf{M}^4 \\ t &\mapsto \mathbf{x}(t), \end{aligned} \tag{2.1}$$

where we use the metric $g = \text{diag}(+ - - -)$. We will not require the normalized tangent vector $\mathbf{v}_1 = d\mathbf{x}/ds$ to be time-like but rather space-like, $\mathbf{v}_1^2 = -1$. This may seem surprising at first but it can be shown [4] that the constraints placed by the dynamics of the rigid particle are only consistent in this regime. The reader may think at this point that this space-like character of the paths will render the theory acausal. That this is not the case can only be understood in terms of the extra gauge invariance of the system. It was shown in [4] how physical (gauge invariant) quantities follow a perfectly consistent standard relativistic motion. We will try to give an intuitive geometric picture of this fact at the end of this section when the reader has already become acquainted with the inner workings of the model.

The extrinsic curvature κ is defined as the modulus of $d\mathbf{v}_1/ds$:

$$\frac{d\mathbf{v}_1}{ds} = \kappa \mathbf{v}_2, \tag{2.2}$$

where \mathbf{v}_2 is orthogonal to \mathbf{v}_1 and, for later consistency with the dynamics, we assume it to be also space-like $\mathbf{v}_2^2 = -1$.

The coordinate expressions of \mathbf{v}_1 , \mathbf{v}_2 and κ are given by

$$\mathbf{v}_1 = \frac{\dot{\mathbf{x}}}{\sqrt{-\dot{\mathbf{x}}^2}}, \quad \mathbf{v}_2 = \frac{\ddot{\mathbf{x}}_\perp}{\sqrt{-\ddot{\mathbf{x}}_\perp^2}}, \quad \kappa = -\frac{\sqrt{-\ddot{\mathbf{x}}_\perp^2}}{\dot{\mathbf{x}}^2} > 0,$$

where $\dot{\mathbf{x}} = d\mathbf{x}/dt$ and $\ddot{\mathbf{x}}_\perp = \ddot{\mathbf{x}} - \dot{\mathbf{x}}(\ddot{\mathbf{x}}\dot{\mathbf{x}})/\dot{\mathbf{x}}^2$.

Now the rigid particle action is defined as the integrated curvature over the worldline:

$$S[\mathbf{x}] = \alpha \int ds \kappa = \alpha \int dt \sqrt{\frac{\ddot{\mathbf{x}}_\perp^2}{\dot{\mathbf{x}}^2}}. \quad (2.3)$$

This is a higher derivative model and we expect its phase space to be larger than the standard cotangent bundle over Minkowski space, which is described solely by the position and total momentum coordinates (\mathbf{x}, \mathbf{P}) . In the case at hand the phase space contains an additional canonical pair $(\dot{\mathbf{x}}, \mathbf{p})$. This can be understood by noting that an arbitrary infinitesimal variation of the action

$$\delta S = - \int_{t_0}^{t_1} dt \dot{\mathbf{P}} \delta \mathbf{x} + \mathbf{P} \delta \mathbf{x} \Big|_{t_0}^{t_1} + \mathbf{p} \delta \dot{\mathbf{x}} \Big|_{t_0}^{t_1}, \quad (2.4)$$

where

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}}, \quad \mathbf{P} = \frac{\partial L}{\partial \ddot{\mathbf{x}}} - \dot{\mathbf{p}},$$

requires not only the equations of motion $\dot{\mathbf{P}} = 0$ to be satisfied with fixed endpoints, but also $\dot{\mathbf{x}}$ should be kept fixed at the endpoints.

Thus, phase space is described by coordinates $(\mathbf{x}, \mathbf{P}, \dot{\mathbf{x}}, \mathbf{p})$ and is endowed with the canonical symplectic form

$$\Omega = d\mathbf{x} \wedge d\mathbf{P} + d\dot{\mathbf{x}} \wedge d\mathbf{p}.$$

The lagrangian expression of the momenta is given by

$$\begin{aligned} \mathbf{p} &= -\frac{\alpha}{\sqrt{-\dot{\mathbf{x}}^2}} \mathbf{v}_2, \\ \mathbf{P} &= \alpha \left(\frac{d\mathbf{v}_2}{ds} + \kappa \mathbf{v}_1 \right). \end{aligned} \quad (2.5)$$

From these expressions and the Frenet equation (2.2) it is easy to show that the \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{P} , form a triad of mutually orthogonal vectors. Moreover, consistency of the equations of motion $d\mathbf{P}/ds = 0$ with the condition $\mathbf{P}\mathbf{v}_2 = 0$ implies that \mathbf{P} has to be a light-like vector.

Indeed,

$$0 = \frac{d\mathbf{P}}{ds}\mathbf{v}_2 + \mathbf{P}\frac{d\mathbf{v}_2}{ds} = \mathbf{P}\left(\frac{\mathbf{P}}{\alpha} - \kappa\mathbf{v}_1\right) = \frac{1}{\alpha}\mathbf{P}^2.$$

All these conditions provide a set of constraints in phase space, $\phi_i \approx 0$, with

$$\begin{aligned}\phi_1 &= \mathbf{p}\dot{\mathbf{x}}, & \phi_2 &= \frac{1}{2}\left(\frac{\dot{\mathbf{x}}^2\mathbf{p}^2}{\alpha} - \alpha\right), \\ \phi_3 &= \mathbf{P}\dot{\mathbf{x}}, & \phi_4 &= \mathbf{P}\mathbf{p}, \\ & & \phi_5 &= \mathbf{P}^2,\end{aligned}\tag{2.6}$$

which turn out to be first-class. It is customary to denote the constraints coming from the definition of the momentum associated with the highest-order time derivative of \mathbf{x} as primary. In our case ϕ_1 and ϕ_2 are the primary constraints and as such they will play an essential role in the reduction process.

First-class constraints generate gauge transformations and the model is certainly invariant under reparametrizations of the worldline. There is, however, an additional gauge symmetry, very peculiar of this model which renders the position of the path as an unphysical (not gauge invariant) quantity. This extra gauge invariance can be given a simple geometrical interpretation as follows [8]. From the curve γ parametrized by $\mathbf{x}(t)$ we can construct a new curve Γ (the Gauss map) which is given by the normalized tangent vector $\mathbf{v}_1(t)$. Then the action (2.3) is nothing more than the arc-length of this new curve:

$$S = \alpha \int dt \sqrt{-\dot{\mathbf{v}}_1^2}.$$

It is clear that there are many different curves sharing the same Gauss map and this can be seen to define the gauge orbits of this extra symmetry. The fact that spacetime trajectories are not physical explains why there should be no a priori inconsistency between the space-like character of the curves and perfectly causal propagation. Indeed, we have explicitly shown how the momentum \mathbf{P} of the particle (which is gauge invariant) has a perfectly well-behaved light-like character.

It was proven in [9] that the full gauge symmetry algebra of the rigid particle is precisely \mathcal{W}_3 . This is most easily done by realising that the equations of motion¹ can be written in terms of the Boussinesq Lax operator. Then, standard methods in integrable systems of the KdV-type show [12] that its symmetry algebra is nothing but the Gel'fand-Dickey bracket associated with $SL(3)$, or equivalently the classical limit of Zamolodchikov's \mathcal{W}_3 -algebra. Therefore establishing a direct connection between the extrinsic geometry of paths in Minkowski, or Euclidean, space-time and \mathcal{W}_3 .

¹ The invariance of the action can be equally checked by purely algebraic methods.

§3 THE COVARIANTLY REDUCED PHASE SPACE OF THE RIGID PARTICLE

We will now perform the covariant reduction of our phase space. We will proceed step by step because on our way some natural mathematical structures, that will be useful in what follows, will surface.

Let us now introduce the following complex coordinates in phase space:

$$\mathbf{z} = \dot{\mathbf{x}} + i \frac{\dot{\mathbf{x}}^2}{\alpha} \mathbf{p}, \quad (3.1)$$

and $\bar{\mathbf{z}}$ its complex conjugate. Notice now that the two primary first-class constraints define a quadric on \mathbf{C}^4 , *i.e.*, $\mathbf{z} \cdot \mathbf{z} = 0$. We can now pass to study the action of these two primary constraints on the quadric, or equivalently, their gauge orbits. A simple computation yields

$$\{\mathbf{z}, \phi_1\} = \mathbf{z}, \quad \text{and} \quad \{\mathbf{z}, \phi_2\} = -i\mathbf{z}. \quad (3.2)$$

Which implies that the flows generated by these two constraints correspond to multiplication by an arbitrary complex number. Therefore if we quotient the phase space with respect to these gauge orbits the reduced phase space (with respect the two primary constraints) is the standard cotangent bundle over Minkowski space-time times a quadric in \mathbf{CP}^3 .

This quadric in \mathbf{CP}^3 has a natural geometric interpretation in terms of the Grassmann manifold of space-like two-planes in Minkowski space, which will be denoted by $G_{(2,4)}^M$. This geometrical identification comes as follows: any space-like two-plane in \mathbf{M}^4 is completely determined by two four-vectors \mathbf{u}_1 and \mathbf{u}_2 which are linearly independent and mutually orthogonal, *i.e.*, $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$. Moreover, without loss of generality one can choose that $|\mathbf{u}_1| = |\mathbf{u}_2|$. Then, if we define $\mathbf{z} = \mathbf{u}_1 + i\mathbf{u}_2$ it follows that $\mathbf{z} \cdot \mathbf{z} = 0$. It is obvious that if we multiply \mathbf{z} by an arbitrary complex number we are still describing the same plane, because the result on the \mathbf{u} 's will simply amount to a combined dilatation and rotation. Equivalently, any \mathbf{z} belonging to this quadric in \mathbf{CP}^3 describes uniquely a space-like plane by choosing $\mathbf{u}_1 = \text{Re } \mathbf{z}$ and $\mathbf{u}_2 = \text{Im } \mathbf{z}$. Notice that the cases in which one or both vectors are time-like or null are directly ruled out.

It is now straightforward to check that the choice of standard inhomogeneous coordinates in the grassmannian, *i.e.*,

$$\mathbf{z} = (1, z^1, z^2, z^3), \quad (3.3)$$

corresponds to the non-covariant gauge-fixing conditions $\dot{x}^0 = 1$ and $p_0 = 0$, which were used by Plyushchay in [4].

Anyhow, we can now proceed in a manifestly covariant manner to compute the symplectic form induced on the grassmannian $G_{(2,4)}^M$ (for the time being we will ignore the term $d\mathbf{x} \wedge d\mathbf{p}$ because it will be irrelevant for this part of the discussion). Notice that

$$d\dot{\mathbf{x}} = \frac{1}{2}(d\mathbf{z} + d\bar{\mathbf{z}}) \quad (3.4)$$

$$d\mathbf{p} = -\frac{i\alpha}{\mathbf{z}\bar{\mathbf{z}}}(d\mathbf{z} - d\bar{\mathbf{z}}) + \frac{i\alpha}{(\mathbf{z}\bar{\mathbf{z}})^2}(\bar{\mathbf{z}}d\mathbf{z} + \mathbf{z}d\bar{\mathbf{z}})(\mathbf{z} - \bar{\mathbf{z}}). \quad (3.5)$$

And from here one obtains

$$d\dot{\mathbf{x}} \wedge d\mathbf{p} = \frac{i\alpha}{(\mathbf{z}\bar{\mathbf{z}})^2} ((\mathbf{z}\bar{\mathbf{z}})d\mathbf{z} \wedge d\bar{\mathbf{z}} - (\bar{\mathbf{z}}d\mathbf{z}) \wedge (\mathbf{z}d\bar{\mathbf{z}})),$$

where we have used in a crucial manner the fact that $\mathbf{z}d\mathbf{z}$ is zero in $G_{(2,4)}^M$. This symplectic form on the grassmannian is precisely the one naturally induced from its embedding in \mathbf{CP}^3 with the standard Fubini-Study symplectic form².

The key observation which will pave our way for the study of the reduced phase space is that points in $G_{(2,4)}^M$ can be understood as complex null lines passing through the origin on \mathbf{C}^4 equipped with a minkowskian metric. This suggests that the appropriate Lorentz invariant formalism is supplied by the standard spinor representation of these null lines. Indeed, the spinor formalism [11] will turn out to be a powerful tool in what follows.

Given an arbitrary complex four-vector \mathbf{y} it can be rewritten in spinor coordinates as follows:

$$(y^{A\dot{U}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} y^0 + y^3 & y^1 + iy^2 \\ y^1 - iy^2 & y^0 - y^3 \end{pmatrix},$$

so that $\det(y^{A\dot{U}}) = \frac{1}{2}g_{\mu\nu}y^\mu y^\nu$.

Because of the two-to-one local isomorphism between $SL(2, \mathbf{C})$ and the identity component of the Lorentz group, one such Lorentz transformation on \mathbf{y} is equivalently represented by the action of an $SL(2, \mathbf{C})$ matrix acting on the undotted indices and its complex conjugate matrix on the dotted ones. Raising and lowering of indices is mimicked in spinor language by contraction with the invariant antisymmetric tensors $(\epsilon^{AB}) = (\epsilon_{AB})$, with $\epsilon^{01} = +1$, and analogous expressions for the dotted indices.

Using the antisymmetry of the invariant tensor ϵ one finds for any (commuting) spinors α and β that $\alpha_A\beta^A = -\alpha^A\beta_A$ and $\alpha_A\alpha^A = 0$, and similarly for dotted spinors.

Coming back to the vector \mathbf{z} defined in (3.1), in spinor coordinates

$$z^\mu \longrightarrow z^{A\dot{U}}, \tag{3.6}$$

and the fact that \mathbf{z} is null, directly implies that

$$z^{A\dot{U}} = \xi^A \bar{\eta}^{\dot{U}}. \tag{3.7}$$

Here ξ and $\bar{\eta}$ are complex spinors, which means eight real degrees of freedom. However, \mathbf{z} is insensitive to a rescaling $\xi \rightarrow a\xi$, $\bar{\eta} \rightarrow \bar{\eta}/a$, with $a \in \mathbf{C}$. Moreover, because of our freedom to rescale \mathbf{z} itself the spinors ξ and $\bar{\eta}$ are both defined only up to an arbitrary complex factor. This again reduces the number of degrees of freedom down to four, in complete agreement with standard hamiltonian counting.

² The reader familiar with the Dirac bracket formalism may be suspicious that we have been oblivious to it. This is not the case, as the Dirac bracket of functions defined on the reduced space certainly coincides with the action of the reduced symplectic form on their associated hamiltonian vector fields. If one chooses to compute Poisson brackets on the constrained surface while still using the \mathbf{z} and $\bar{\mathbf{z}}$ coordinates, the gradients of the associated functions turn out to be ill-defined. By imposing on them that their hamiltonian vector fields be tangent to the constrained manifold, one easily recovers the standard Dirac bracket formalism [13].

It is then a direct computation to check that the symplectic form in spinor coordinates is given by

$$-i\alpha \frac{d\eta_A \wedge d\xi^A}{\xi^C \eta_C} + i\alpha \frac{\eta_A \xi^B d\eta_B \wedge d\xi^A}{(\xi^C \eta_C)^2} + c.c. \quad (3.8)$$

Notice now that the three remaining constraints (ϕ_3, ϕ_4, ϕ_5) that we have been ignoring so far, can be neatly written in spinor form. On the one hand the fact that $\mathbf{P}^2 = 0$ implies that \mathbf{P} is a real null vector and hence can be written as

$$P^\mu \longrightarrow P^{A\dot{A}} = \pm \pi^A \bar{\pi}^{\dot{A}}, \quad (3.9)$$

where π is completely determined up to an arbitrary phase factor, and the plus and minus signs correspond to future or past pointing null vectors respectively. And on the other hand

$$\mathbf{P}\mathbf{z} = 0 \longrightarrow (\pi_A \xi^A)(\bar{\pi}_{\dot{A}} \bar{\eta}^{\dot{A}}) = 0, \quad (3.10)$$

so in its spinor form this constraint reduces to either

$$\pi_A \xi^A = 0 \quad \text{or} \quad \bar{\pi}_{\dot{A}} \bar{\eta}^{\dot{A}} = 0. \quad (3.11)$$

First notice that both conditions cannot be simultaneously fulfilled because it contradicts the two primary constraints ϕ_1 and ϕ_2 . In fact, if both conditions were obeyed then \mathbf{z} would be proportional to \mathbf{P} and therefore not only $\mathbf{z}^2 = 0$ but also $\mathbf{z}\bar{\mathbf{z}} = 0$, which would require $\dot{\mathbf{x}}$ to be null in clear contradiction to ϕ_2 . Therefore the reduced phase space has four different branches that will be denoted by M_α^+ , $M_{-\alpha}^+$, M_α^- and $M_{-\alpha}^-$, where the superscript $+$ ($-$) corresponds to future (past) pointing momentum, and the subscripts $\pm\alpha$ correspond to different values of the helicity that will correspond, as we will show below, to the two possible choices between the spinor constraints (3.11).

Let us recall that the (physical) irreducible representations of the Poincaré algebra are labeled by the values of the two casimirs $\mathbf{P}^2 = m^2$ and the square of the Pauli-Lubanski vector

$$S^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}. \quad (3.12)$$

In the massless case, if one disregards the unphysical situation when $\mathbf{S}^2 \neq 0$, it directly follows that $\mathbf{S} = s\mathbf{P}$ for some s . This invariant is usually denoted as the helicity. We can now show that our case will fall under this category.

First from the constraint $\mathbf{P}^2 = 0$ it directly follows that we are dealing with the massless case. Moreover, the “internal” angular momentum matrix is given by

$$M^{\mu\nu} = \dot{x}^{[\mu} p^{\nu]} = -\frac{i\alpha}{\mathbf{z}\bar{\mathbf{z}}} \bar{z}^{[\mu} z^{\nu]}. \quad (3.13)$$

And from this it follows that

$$S_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} = -\frac{i\alpha}{\mathbf{z}\bar{\mathbf{z}}} \epsilon_{\mu\nu\rho\sigma} P^\nu \bar{z}^\rho z^\sigma. \quad (3.14)$$

In spinor language the above expression reads

$$S_{A\dot{A}} = \pm \frac{\alpha}{(\xi\eta)(\bar{\xi}\bar{\eta})} (\epsilon_{AD}\epsilon_{BC}\epsilon_{\dot{A}\dot{C}}\epsilon_{\dot{B}\dot{D}} - \epsilon_{AC}\epsilon_{BD}\epsilon_{\dot{A}\dot{D}}\epsilon_{\dot{B}\dot{C}}) \pi^B \bar{\pi}^{\dot{B}} \bar{\xi}^{\dot{C}} \eta^C \xi^D \bar{\eta}^{\dot{D}}. \quad (3.15)$$

Note that if we choose future pointing \mathbf{P} and the branch of the constraint surface given by

$\pi_A \xi^A = 0$ it follows that

$$S_{A\dot{A}} = \alpha \pi_A \bar{\pi}_{\dot{A}}, \quad (3.16)$$

or equivalently $\mathbf{S} = s\mathbf{P}$ with helicity $s = \alpha$. But if the other branch $\pi_A \eta^A$ had been chosen we would have got a similar result, but now with a value $-\alpha$ for the helicity. The past pointing \mathbf{P} can be similarly worked out, thus justifying our notational choice.

If one considers the whole Poincaré group and not only its connected component, it can be shown [10] that a Lorentz transformation preserving spatial orientation, but reversing the arrow of time, maps M_s^+ into M_{-s}^- ; and maps M_s^+ into M_s^- whenever it reverses both space and time orientations. It is therefore natural to identify those subspaces and one can regard the phase space as the union of M_s^+ and M_{-s}^+ . This can be easily understood in our model because time reversal maps the equivalence class of \mathbf{z} into the one of $\bar{\mathbf{z}}$ thus interchanging the two possible branches of our constraint (3.11).

Now we come back to our reduced phase space. We recall that ξ and η were both defined up to a multiplication by an arbitrary complex number. On the constraint surface given by $\pi_A \xi^A = 0$ the spinor ξ must be proportional to π , so we can remove the above freedom in ξ and η by setting

$$\xi^A = \pi^A \quad \text{and} \quad \eta^A \pi_A = 1, \quad (3.17)$$

i.e., π and η form a spinor basis. Notice also that because of the phase ambiguity in π we have an equivalent ambiguity left in η . This corresponds to a reduction of the subspace previously denoted by M_α^+ if one chooses future pointing \mathbf{P} .

One can now compute the induced symplectic form on the submanifold defined by (3.17). One readily obtains:

$$\pi_A dx^{A\dot{A}} \wedge d\bar{\pi}_{\dot{A}} + i\alpha d\eta_A \wedge d\pi^A + c.c. \quad (3.18)$$

It follows from its definition that the above form is degenerate. This is what is to be expected in the case of first-class constraints unless one introduces enough gauge conditions to turn all of them into second class. This is indeed the case here. We have already solved all the constraints ϕ_1, \dots, ϕ_5 but have only performed the quotient over the orbits generated by ϕ_1 and ϕ_2 . Therefore before continuing one should identify which vectors are in the kernel of this form, *i.e.* determine the vector fields tangent to the remaining orbits. With a little of hindsight due to the particular structure of the constraints it is simple to check that

$$X_1 = \pi^A \bar{\pi}^{\dot{A}} \frac{\partial}{\partial x^{A\dot{A}}}, \quad (3.19)$$

$$X_2 = \pi_A \frac{\partial}{\partial \eta_A} + i\alpha \bar{\eta}^{\dot{A}} \pi^A \frac{\partial}{\partial x^{A\dot{A}}}, \quad (3.20)$$

$$X_3 = i\eta^A \frac{\partial}{\partial \eta^A} - i\pi_A \frac{\partial}{\partial \pi_A} + c.c.$$

span the kernel of (3.18). Notice that X_2 is complex so the real dimension of the space spanned by X_1 and X_2 is three, in full agreement with the dimension of the orbits generated by ϕ_3, ϕ_4 and ϕ_5 . The vector field X_3 is responsible for the phase shifts in π and η . With

the above result in mind one could directly apply the general reduction procedure of [14], however things are greatly simplified by realising that π_A and ω^A with

$$\omega^A = \alpha \eta^A + ix^{A\dot{A}} \bar{\pi}_{\dot{A}}$$

are constant along the orbits generated by X_1 and X_2 . So they are natural coordinates to describe the fully reduced phase space. From all of this it follows that M_α^+ can be parametrized by the four components of the two spinors ω and $\bar{\pi}$ subject to the equivalence relation

$$(\omega, \bar{\pi}) \equiv (e^{i\beta} \omega, e^{i\beta} \bar{\pi})$$

with $\beta \in \mathbf{R}$ and obeying

$$\omega^A \pi_A + \bar{\omega}^{\dot{A}} \bar{\pi}_{\dot{A}} = 2\alpha, \quad (3.21)$$

which is just the constraint $\eta^A \pi_A = 1$ in (3.17) written in the (ω, π) variables. The symplectic form in these variables can be directly read from (3.18), and is given by

$$\Omega = -id\omega^A \wedge d\pi_A + c.c. \quad (3.22)$$

The kernel of the symplectic form induced on the constrained surface defined by the first-class constraint (3.21) precisely generates the phase shifts in ω and $\bar{\pi}$. So if \mathbf{T} (for twistor space [11]) denotes the four dimensional complex vector space on which ω^A and $\bar{\pi}_{\dot{A}}$ are independent linear coordinates it follows that M_α^+ is nothing but the reduction of \mathbf{T} with respect to the first-class constraint (3.21).

It is a standard result from the theory of coadjoint orbits that the above phase space can be identified with the coadjoint orbit of the Poincaré group associated with massless particles with helicity α and future pointing momentum, with Ω being the Kirillov-Kostant symplectic structure associated with those orbits.

A completely analogous analysis can be carried out for $M_{-\alpha}^+$ yielding a similar result up to the relative sign of the helicity. The identification with the associated Poincaré orbits is, of course, maintained.

Due to the relationship of these orbits with twistor space one can give an alternative description of them in twistor variables as follows [15]. If Z represents the pair of spinors $(\omega, \bar{\pi})$, then one can take as twistor components

$$Z^\gamma = (\omega^0, \omega^1, \bar{\pi}_{\dot{0}}, \bar{\pi}_{\dot{1}}). \quad (3.23)$$

If one defines the conjugate twistor \bar{Z} by

$$\bar{Z}_\gamma = (\pi_0, \pi_1, \bar{\omega}^{\dot{0}}, \bar{\omega}^{\dot{1}}), \quad (3.24)$$

the constraint (3.21) can be simply written as $Z^\gamma \bar{Z}_\gamma = 2\alpha$. Finally the symplectic form can be written as

$$-idZ^\gamma \wedge d\bar{Z}_\gamma + c.c. \quad (3.25)$$

With all of this in mind we will now pass to quantize the rigid particle model.

§4 QUANTIZATION

Due to the identification of the reduced phase space of the rigid particle model with the coadjoint orbits of the Poincaré group for massless particles with helicity s , the quantization of this system is an exercise that has already been the object of study in standard textbooks. It will be certainly out of the scope of this paper to give a full account of the standard procedures, and for that we will refer the reader to the excellent book of Woodhouse on geometric quantization [10]. Anyhow, as the full machinery of geometric quantization is not entirely necessary to understand the quantization of such a simple system, we will attempt here to extract from [10] the bare essentials.

The covariant quantization of the model will now be performed *à la* Dirac, by imposing the first-class constraint (3.21) on the physical states. First we start by choosing a polarization generated by $\partial/\partial\omega$ and its complex conjugate, *i.e.*, we will choose our wave functions to be functions of π and $\bar{\pi}$. In this representation it is obvious that the operator associated with ω becomes

$$\omega^A \rightarrow -\frac{\partial}{\partial\pi_A}, \quad \bar{\omega}^{\dot{A}} \rightarrow \frac{\partial}{\partial\bar{\pi}_{\dot{A}}}, \quad (4.1)$$

while the operator associated with π is simply given by multiplication by π .

The constraint (3.21) can be now directly written as

$$\bar{\pi}_{\dot{A}} \frac{\partial}{\partial\bar{\pi}_{\dot{A}}} - \pi_A \frac{\partial}{\partial\pi_A} = 2\alpha. \quad (4.2)$$

It is easy to show that this expression does not suffer from ordering ambiguities provided that we choose the same ordering for the two pairs (π, ω) and $(\bar{\pi}, \bar{\omega})$, since the right-hand-side of their respective commutators have opposite signs and hence the ordering ambiguities cancel.

The quantization of α can now be understood in several ways. The more geometrical one is associated with the integrality condition of the symplectic potential

$$\theta = -i\omega^A d\pi_A + c.c. \quad (4.3)$$

in the constrained manifold. But it can also be understood in a more standard physical way by showing the equivalence of the associated Hilbert space with the solution space of massless wave equations of arbitrary spin. Indeed, for positive helicity, the wave functions $\varphi(\pi, \bar{\pi})$ obeying the constraint (4.2) can be mapped consistently into the positive frequency solutions of the left-handed massless wave equation [11]

$$\nabla^{A\dot{A}} \varphi_{AB\dots C}(\mathbf{x}) = 0 \quad (4.4)$$

by means of the Fourier transform

$$\varphi_{AB\dots C}(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{N^+} \varphi(\pi, \bar{\pi}) \pi_A \pi_B \dots \pi_C e^{-i\mathbf{P}\mathbf{x}} d\tau, \quad (4.5)$$

where there are 2α π 's, and the integration is over the future light cone with $d\tau$ its natural

Lorentz invariant measure

$$d\tau = \frac{dP_1 \wedge dP_2 \wedge dP_3}{|P_0|}. \quad (4.6)$$

Because of the homogeneity properties of $\varphi(\pi, \bar{\pi})$ the integrand is well defined in the constrained surface, *i.e.*, it is impervious to transformations of the form $\pi \rightarrow e^{i\beta} \pi$ for $\beta \in \mathbf{R}$.

The right-handed solutions can be equally obtained by recalling the natural correspondence between $M_{-\alpha}^+$ and M_{α}^- .

A quantization in the twistor polarisation has been already pursued in simple terms in [15]. The interested reader can find there all the required information, so we will avoid here any unnecessary repetition.

We would like to stress, as a final remark, that the conformal invariance of these massless spin equations has a natural counterpart in the particle model. It is evident from the definition of the action (1.1) that it only depends on the conformal class of the Minkowski metric.

In the search for gauge invariance

Although from the quantization of the Poincaré orbit for $\alpha = 1$ one obtains directly Maxwell equations in spinor form, as a physicist, one is a little disappointed by the fact that the gauge potential does not seem to come out from the formalism. As we will see below not only the gauge potential is naturally there, but we will be able to interpret its associated $U(1)$ gauge transformations as a direct consequence of the constraint structure of the model arising from its W_3 symmetry.

In order to see how this happens one should return to the (pre)symplectic two-form in spinor coordinates (3.8)

$$-i\alpha \frac{d\eta_A \wedge d\xi^A}{\xi^C \eta_C} + i\alpha \frac{\eta_A \xi^B d\eta_B \wedge d\xi^A}{(\xi^C \eta_C)^2} + c.c. \quad (4.7)$$

The key observation is that we can greatly simplify its expression after a suitable redefinition of constraints. Indeed, notice that the two-form (4.7) reduces to

$$-id\eta_A \wedge d\xi^A + c.c. \quad (4.8)$$

on the submanifold defined by the first-class constraint $\xi^A \eta_A = \alpha$. That one can impose consistently this condition follows directly from the fact that ξ and η are defined only up to multiplication by an arbitrary complex number, and that $\xi^A \eta_A$ cannot be zero (otherwise $\mathbf{z} \cdot \bar{\mathbf{z}} = 0$). It can also be easily checked that the kernel of (4.8) on the constrained submanifold generates the remaining freedom left in the spinors. The constraint above reduces the arbitrariness in the spinors to $\xi \rightarrow a\xi$ and $\eta \rightarrow (1/a)\eta$. One can check now that exactly those gauge orbits are the ones generated by the hamiltonian vector fields associated with the constraint and its complex conjugate.

The way to recover the gauge potential should be by now clear: one should quantize the phase space with coordinates $(\mathbf{x}, \mathbf{P}, \eta^A, \xi^A)$, symplectic form

$$d\mathbf{x} \wedge d\mathbf{P} - i d\eta_A \wedge d\xi^A + c.c. \quad (4.9)$$

and subject to the first-class constraints³

$$\psi_1 = \eta_A \xi^A - \alpha, \quad \psi_2 = P_{B\dot{B}} \xi^B \bar{\eta}^{\dot{B}}, \quad \text{and} \quad \psi_3 = \mathbf{P}^2,$$

with Dirac's prescription for first-class constraints.

Explicitly, if one takes the natural polarization associated with the symplectic potential

$$\theta = -\mathbf{P} d\mathbf{x} - i\eta_A d\xi^A - i\bar{\xi}_{\dot{A}} d\bar{\eta}^{\dot{A}}, \quad (4.10)$$

one has that under quantization

$$\mathbf{x} \rightarrow -i \frac{\partial}{\partial \mathbf{P}}, \quad \xi_A \rightarrow \frac{\partial}{\partial \eta^A}, \quad \text{and} \quad \bar{\eta}_{\dot{A}} \rightarrow \frac{\partial}{\partial \bar{\xi}^{\dot{A}}}, \quad (4.11)$$

while \mathbf{P} , η and $\bar{\xi}$ go to standard multiplication operators acting on wave functions $A(\mathbf{P}, \eta, \bar{\xi})$.

If we now impose the constraints as operator identities on the wave functions we obtain

$$\eta^A \frac{\partial}{\partial \eta^A} \cdot A(\mathbf{P}, \eta, \bar{\xi}) = \alpha A(\mathbf{P}, \eta, \bar{\xi}), \quad (4.12)$$

$$\bar{\xi}^{\dot{A}} \frac{\partial}{\partial \bar{\xi}^{\dot{A}}} \cdot A(\mathbf{P}, \eta, \bar{\xi}) = \alpha A(\mathbf{P}, \eta, \bar{\xi}), \quad (4.13)$$

$$P_{A\dot{A}} \eta^A \bar{\xi}^{\dot{A}} A(\mathbf{P}, \eta, \bar{\xi}) = 0, \quad (4.14)$$

$$P^{A\dot{A}} \frac{\partial}{\partial \eta^A} \frac{\partial}{\partial \bar{\xi}^{\dot{A}}} \cdot A(\mathbf{P}, \eta, \bar{\xi}) = 0, \quad (4.15)$$

$$P^2 A(\mathbf{P}, \eta, \bar{\xi}) = 0. \quad (4.16)$$

Notice that in this case the quantization of the constraints ψ_1 and $\bar{\psi}_1$ suffers from ordering ambiguities and here we have chosen to write all derivative operators on the right; the only choice consistent with the covariantly reduced space quantization of the previous section. These ordering ambiguities however do not affect the first-class character of the quantum constraint algebra.

The first two constraints for $\alpha = 1$ simply tell us that the wave function is of the form

$$A(\mathbf{P}, \eta, \bar{\xi}) = A_{B\dot{B}}(\mathbf{P}) \eta^B \bar{\xi}^{\dot{B}}. \quad (4.17)$$

But notice from the third condition that $A_{B\dot{B}}(\mathbf{P})$ is only defined up to a term of the form $P_{B\dot{B}} \varphi(\mathbf{P})$, and this is nothing but the standard gauge transformation of the vector potential in momentum space. The remaining two constraints can be now seen to impose the Lorenz gauge condition and the mass shell condition respectively.

³ Notice that due to the complex character of ψ_1 and ψ_2 the counting of the number of degrees of freedom yields the correct result.

The above result can be given a clear geometrical interpretation. The gauge invariance of the wave functions comes as a direct consequence of the fact that the trajectories in the rigid particle model are not physical. This agrees with the intuitive idea that gauge invariance improves the renormalizability properties of the associated quantum theory by delocalizing the position of the photon. The rigid particle model provide us with a precise mathematical description of this intuitive physical idea.

The details of the generalization of the above procedure for other integer values of α will be left as an exercise for the interested reader. It is obvious from the previous construction that only the particular form of the wave functions and their corresponding gauge invariances will depend on the particular value of α , not so the quantization procedure sketched above. We will finish this section by stating that, in particular, for $\alpha = 2$ one obtains linearized Einstein gravity in terms of the (traceless) metric deviation from flat space-time in the Einstein gauge. In this case the corresponding gauge invariance is nothing but linearized general covariance.

§5 FINAL COMMENTS

We hope to have convinced the reader that the rigid particle model and its associated W_3 symmetry play an important role in the physics of massless particle models in four dimensions, as well as in the geometry of gauge invariance. It is also, in our opinion, quite remarkable that a purely “bosonic” particle model is suitable for the description of Weyl fermions in four dimensional Minkowski space-time. Thus opening the door to the understanding of four-dimensional Fermi-Bose transmutation in terms of this system (for a somehow related approach see [16][17]).

It is natural to wonder if a similar approach can be developed for the massive case. Unfortunately, the adding of an explicit mass term to the rigid particle model leads to a reduced phase space without the adequate dimensions [18]. It is therefore an open problem to find a geometrical particle model which can be naturally associated with massive particles of arbitrary spin.

It would be also interesting to investigate if more general geometrical particle models can incorporate naturally, under quantization, a bigger gauge invariance group than $U(1)$. Under the condition of locality and invariance under conformal rescaling of the metric, a property that should be preserved if one wishes to obtain conformally invariant field theories, the most general four dimensional particle action is of the form

$$S = \sum_{i=1}^3 \alpha_i \int_{\gamma} \kappa_i, \quad (5.1)$$

where the κ ’s correspond to the generalized curvature functions associated with the path γ . The required phase space has dimension 32, although of course a plethora of constraints will naturally arise from (5.1). The structure of the reduced phase space for particular values of the parameters α_i can a priori host non-abelian gauge invariance—a possibility that is under current investigation.

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